COBEM 99
XV Congresso Brasileiro de Engenharia Mecânica
15th Brazilian Congress of Mechanical Engineering
22 - 26 de Novembro de 1999 / November 22 - 26, 1999 Águas de Lindóia, São Paulo,

ROBUSTLY CONTRACTIVE POLYHEDRA FOR UNCERTAIN LINEAR SYSTEMS WITH SATURATING CONTROLS

Basílio E. A. Milani
Alessandra D. Coelho
Wânderson O. Assis
UNICAMP, Faculdade de Engenharia Elétrica e de Computação
Cx. P. 6101 - 13081-970 - Campinas, SP, Brasil

Abstract. *This paper presents new necessary and sufficient conditions for compact polyhedral sets to be robustly λ-contractive with respect to uncertain discrete-time linear systems with saturating control inputs. Based on linear programming formulation of these conditions, an effective procedure is proposed for construction of robustly λ-contractive polyhedral sets with nonempty intersection with the region of nonlinear behavior of the closed-loop system. The procedure starts with the supremal robustly λ-contractive set contained in the region of linear behavior and progressively expands it over the region of nonlinear behavior.*

Key Words: *Uncertain linear systems; Discrete-time systems, Robustly contractive sets; Saturating controls; Linear programming.*

1. INTRODUCTION

Consider an uncertain autonomous dynamical system represented by a family of discrete time models $\mathcal{S}$ (Barmish, 1994). Roughly speaking, a nonempty set Ω defined in state space of $\mathcal{S}$, is said to be robustly λ-contractive with respect to (w.r.t.) $\mathcal{S}$ if there is a real $0 \leq \lambda < 1$ such that: for all $\mathcal{S}$ members, if state $x(k) \in \epsilon\Omega$ then $x(k+1) \in \lambda\epsilon\Omega$ for all $0 < \epsilon \leq 1$. Robustly contractive sets are important in analysis and control design for both linear and nonlinear uncertain systems: they correspond to regions of robust asymptotic stability and they can also play the role of "confinement sets", a key idea widely exploited in the case of systems subject to constraints (Milani & Carvalho, 1995), (Verriest & Pajunen, 1996), (Blanchini, 1994).

When designing stabilizing feedback controllers for linear systems subject to control bounds, a prevalent and challenging aspect to be considered is nonlinearity due to saturation of control inputs. A common but conservative approach is to search for λ-contractive sets that avoid saturation, keeping the closed-loop system working in its region of linear behavior (Vassilaki et al., 1988), (Gilbert & Tan, 1991). A more effective, but also more difficult approach is to allow saturation of control inputs and consequent nonlinear behavior of the closed-loop system. In this sense, for discrete-time systems with perfectly known models, Silva Jr. & Tarbouriech (1997), using a piecewise linear description of the closed-loop system, present a necessary and sufficient condition for convex polyhedral sets be λ-contractive. The proposed condition requires the knowledge of the extreme points associated to each face of the polyhedral set, which complicates its use even in problems of moderate dimensions and discourages its extension to uncertain systems.

This paper deals with the characterization and construction of robustly λ-contractive sets w.r.t. uncertain discrete-time linear systems with saturating control inputs. New necessary and sufficient conditions are proposed for compact polyhedral sets be robustly λ-contractive w.r.t. discrete-time systems with uncertain domains defined by compact polytopes. Based on linear programming formulation of these necessary and sufficient conditions, an effective non homothetic expansion procedure is proposed for construction of robustly λ-contractive sets with nonempty intersection with regions of closed-loop nonlinear behavior of uncertain systems.

Throughout this paper, for two $n \times m$ real matrices $A = (a_{ij})$ and $B = (b_{ij})$, $A \leq B$ is equivalent to $a_{ij} \leq b_{ij}$ for all i, j such that $1 \leq i \leq n$ and $1 \leq j \leq m$. $A \geq 0$ is equivalent to $a_{ij} \geq 0$ and for any real $\epsilon \geq 0$, the set $\epsilon\Omega$ is defined as $\{x = \epsilon y, y \in \Omega\}$.

2. SATURATING FEEDBACK CONTROL MODEL

Consider the discrete-time uncertain linear system represented by the following state equations and constraints:

$$x(k+1) = Ax(k) + Bu(k) \;\; ; \;\; (A,B) \in \mathcal{P} \tag{1}$$

$$-\check{u} \leq u \leq \hat{u} \;\; ; \;\; \check{u}, \hat{u} \geq 0 \tag{2}$$

where $x(k) \in \Re^n$, $u(k) \in \Re^m$, are state and control variables respectively and matrix pair (A, B) is unknown but restricted to set $\mathcal{P} \subset (\Re^{n \times n} \times \Re^{n \times m})$. Assume the saturating feedback control law

$$u(k) = sat(Fx(k)) \tag{3}$$

where $F \in \Re^{m \times n}$ is constant and the components of $sat(Fx)$ are given by:

$$sat(Fx)_i = \begin{cases} -\check{u}_i & if \;\; f_i x < -\check{u}_i \\ f_i x & if \;\; -\check{u}_i \leq f_i x \leq \hat{u}_i \\ \hat{u}_i & if \;\; f_i x > \hat{u}_i \end{cases} \tag{4}$$

where f_i denotes the ith row of matrix F.

From Eq. (1), (3), the closed-loop system is given by the nonlinear model:

$$x(k+1) = Ax(k) + Bsat(Fx(k)) \;\; ; \;\; (A,B) \in \mathcal{P} \tag{5}$$

Considering all $x \in \Re^n$, each one of the m components of the saturating law Eq. (4) has 3 possible states: saturated at lower bound, not saturated and saturated at upper bound. Consequently, $\Re^n$ can be decomposed into $j = 1 : 3^m$ regions $S(R_j, d_j) \subset \Re^n$, called saturation regions (Silva Jr. & Tarbouriech, 1997), given by polyhedra:

$$S(R_j, d_j) = \{x \in \Re^n; R_j x \leq d_j\} \tag{6}$$

$$R_j = \begin{bmatrix} F_{ns} \\ -F_{ns} \\ -F_{su} \\ F_{sl} \end{bmatrix} \;\; ; \;\; d_j = \begin{bmatrix} \hat{u}_{ns} \\ \check{u}_{ns} \\ -\hat{u}_{su} \\ -\check{u}_{sl} \end{bmatrix} \tag{7}$$

where F_{ns}, $\hat{u}_{ns}$, $\check{u}_{ns}$, F_{su}, $\hat{u}_{su}$, F_{sl}, $\check{u}_{sl}$, denote matrices and vectors appropriately formed by the rows of F, $\hat{u}$, $\check{u}$, related, respectively, to the components not saturated, saturated at upper level and saturated at lower level, which characterize the region. Within each saturation region $S(R_j, d_j)$, closed-loop system Eq. (5) is represented by a linear model of the form (Silva jr. & Tarbouriech, 1997):

$$\begin{array}{rl} x(k+1) = A_j x(k) + p_j & \\ A_j = [A + B_{ns}F_{ns}] & ; \ (A,B) \in \mathcal{P} \\ p_j = B_{su}\hat{u}_{su} - B_{sl}\check{u}_{sl} & \end{array} \tag{8}$$

where B_{ns}, B_{su} and B_{sl} denote matrices appropriately formed by the columns of B related to F_{ns}, $\hat{u}_{su}$, $\check{u}_{sl}$, respectively. Throughout the paper, it will be assigned $j = 1$ for the region of linear behavior of $sat(F(x))$, described by:

$$R_1 = \begin{bmatrix} F \\ -F \end{bmatrix} \ ; \ d_1 = \begin{bmatrix} \hat{u} \\ \check{u} \end{bmatrix} \ ; \ A_1 = A + BF \ ; \ p_1 = 0 \ ; \ (A,B) \in \mathcal{P} \tag{9}$$

3. ROBUSTLY CONTRACTIVE SETS

Consider closed-loop system, Eq. (5), and the convex compact polyhedron:

$$S(G,w) = \{x \in \Re^n ; Gx \leq w\} \tag{10}$$

where $G \in \Re^{r \times n}$ and $w > 0 \in \Re^r$.

Definition 1: Polyhedron $S(G,w)$, Eq. (10), is said to be robustly λ-contractive w.r.t. uncertain closed-loop system, Eq. (5), if there is a real $0 < \lambda < 1$ such that $x(k+1) \in S(G, w\epsilon\lambda)$ for all $0 < \epsilon \leq 1$, all $x(k) \in S(G, w\epsilon)$ and all $(A,B) \in \mathcal{P}$.

Definition 2: The one-step robustly admissible set to $S(G,w)$ w.r.t. uncertain system, Eq. (5), is given by:

$$Q(G,w) = \{x \in \Re^n ; G[Ax + Bsat(F(x))] \leq w \ \ \forall \ \ (A,B) \in \mathcal{P}\} \tag{11}$$

Proposition 1: Polyhedron $S(G,w)$, Eq. (10), is robustly λ-contractive w.r.t. uncertain closed-loop system, Eq. (5), iff there is a real $0 < \lambda < 1$ such that:

$$S(G, \epsilon w) \subset Q(G, \lambda\epsilon w) \ \ \forall \ \ 0 < \epsilon \leq 1 \tag{12}$$

Proof: immediate from Definition 1. □

Let uncertain domain $\mathcal{P}$ be given by convex compact polytope:

$$\mathcal{P} = \left\{(A,B) : \ (A,B) = \sum_{l=1}^{l=s} \xi_l (A^l, B^l) \ ; \ \xi_l \geq 0 \ ; \ \sum_{l=1}^{l=s} \xi_l = 1 \right\} \tag{13}$$

where $(A^l, B^l) \in \Re^{n \times n} \times \Re^{n \times m}$ are extreme points.

For $\mathcal{P}$ given by Eq. (13), noting that $G[Ax + Bsat(F(x)] \leq w$ is affine in (A,B), it is easy to verify that $Q(G,w)$ is given by

$$Q(G,w) = \{x \in \Re^n ; G[A^l x + B^l sat(F(x))] \leq w \ \ l = 1:s\} \tag{14}$$

Moreover, from Eq. (6) to (8), it is also easy to verify that $Q(G,w)$ restricted to saturation region $S(R_j, d_j)$ is given by

$$Q_j(G,w) = \{x \in \S(R_j, d_j) ; G[A_j^l x + p_j^l] \leq w \ \ l = 1:s\} \tag{15}$$

where A_j^l, p_j^l, are the linear model parameters in $S(R_j, d_j)$ related to extreme point (A^i, B^i).

Based on Proposition 1 and Eq. (14), (15), the following Proposition gives a necessary and sufficient condition for a polyedral set be robustly λ-contractive.

Proposition 2: Considering uncertain parameter domain, Eq. (13), polyhedron $S(G, w)$, Eq. (10), is robustly λ-contractive w.r.t. closed-loop system, Eq. (5), iff for the $j = 1 : 3^m$ saturation regions Eq. (6), (7), (8) and for the $l = 1 : s$ extreme points of $\mathcal{P}$, there is a real $0 < \lambda < 1$ such that

$$\begin{bmatrix} GA_j^l & -\lambda w \end{bmatrix} \begin{bmatrix} x \\ \epsilon \end{bmatrix} \leq -Gp_j^l \tag{16}$$

holds for any ϵ, x satisfying:

$$\begin{bmatrix} G & -w \\ R_j & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ \epsilon \end{bmatrix} \leq \begin{bmatrix} 0 \\ d_j \\ 1 \\ 0 \end{bmatrix} \tag{17}$$

or, in other words, iff there is $0 < \lambda < 1$ such that polyhedron Eq. (17) is a subset of polyhedra Eq. (16).

Proof: From Proposition 1, $S(G, w)$ is robustly λ-contractive iff

$$S(G, \epsilon w) \subset Q(G, \lambda \epsilon w) \;\; \forall \;\; 0 < \epsilon \leq 1 \tag{18}$$

Equations

$$\Re^n = \bigcup_{j=1}^{j=3^m} S(R_j, d_j) \tag{19}$$

$$S(G, \epsilon w) = S(G, \epsilon w) \cap \Re^n \tag{20}$$

give

$$S(G, \epsilon w) = \bigcup_{j=1}^{j=3^m} S(G, \epsilon w) \cap S(R_j, d_j) \tag{21}$$

From Eq. (15), (18), (21), it can be verified that

$$S(G, \epsilon w) \cap S(R_j, d_j) \subset Q_j(G, \lambda \epsilon w) \;\; \forall \;\; 0 < \epsilon \leq 1 \;\; j = 1 : 3^m \tag{22}$$

From Eq. (15), $Q_j(G, \lambda \epsilon w)$ is a polyhedron given by the intersection of $l = 1 : s$ polyhedra:

$$\begin{bmatrix} GA_j^l & -\lambda w \end{bmatrix} \begin{bmatrix} x \\ \epsilon \end{bmatrix} \leq -Gp_j^l \tag{23}$$

From Eq. (6), (7), (10), $S(G, \epsilon_j w) \cap S(R_j, d_j)$ is given by polyhedron:

$$\begin{bmatrix} G & -w \\ R_j & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ \epsilon_j \end{bmatrix} \leq \begin{bmatrix} 0 \\ d_j \\ 1 \\ 0 \end{bmatrix} \tag{24}$$

It is easy to verify that inclusion relation, Eq. (22), holds iff polyhedron, Eq. (24), is a subset of all polyhedra in Eq. (23), which concludes the proof. □.

The following Corollary gives a linear programming formulation to Proposition 2.

Corollary 1: Polyhedron $S(G, w)$, Eq. (10), is robustly λ-contractive w.r.t. closed-loop system, Eq. (5), iff there is a real $0 < \lambda < 1$ such that:

$$\max_{j,l,i}\{\sigma(j)_l^i\} \leq 0 \tag{25}$$

$$1 \leq j \leq 3^m \;\; ; \;\; 1 \leq l \leq s \;\; ; \;\; 1 \leq i \leq r$$

such that $\sigma(j)_l^i$ are obtained solving the following independent feasible linear programs:

$$\sigma(j)_l^i = \max_{x,\epsilon}(g^i A_j^l x - \lambda w_i \epsilon + g^i p_j^l)$$

subject to:

$$\begin{array}{c} Gx - w\epsilon \leq 0 \\ R_j x \leq d_j \\ 0 \leq \epsilon \leq 1 \end{array} \tag{26}$$

where g^i, w_i, are the ith row of G, w, respectively and R_j, d_j, A_j^l, p_j^l are related to saturation regions, piecewise linear model Eq. (6) to (8) and uncertain parameter domain Eq. (13). Furthermore, let $\epsilon(j)_l^i$, $x(j)_l^i$ be an optimal solution related to $\sigma(j)_l^i > 0$. This indicates that $x(j)_l^i \in S(G, \epsilon(j)_l^i w)$ is outside the ith face of $Q_j(G, \lambda w \epsilon(j)_l^i)$, Eq. (15):

$$g^i A_j^l x - \lambda w_i \epsilon(j)_l^i \leq -g^i p_j^l \tag{27}$$

Proof: Inspecting Eq. (16), (17), it is easy to verify that Proposition 2 is satisfied iff

$$\begin{array}{c} g^i A_j^l x - \lambda w_i \epsilon \leq -g^i p_j^l \\ j = 1 : 3^m \;\; ; \;\; i = l : s; \;\; i = 1 : r \end{array} \tag{28}$$

hold for all ϵ, x satisfying Eq. (17). In other words,

$$\max_{x,\epsilon}(g^i A_j^l x - \lambda w_i \epsilon + g^i p_j^l) \leq 0 \tag{29}$$

subject to:

$$\begin{bmatrix} G & -w \\ R_j & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ \epsilon \end{bmatrix} \leq \begin{bmatrix} 0 \\ d_j \\ 1 \\ 0 \end{bmatrix}$$

must hold for $j = 1 : 3^m$, $l = 1 : s$ and $i = 1 : r$. It is easy to verify that linear programs Eq. (29) are equivalent to linear programs Eq. (25), (26). Let $\sigma(j)_l^i$ and $\epsilon(j)_l^i$, $x(j)_l^i$, denote respectively, optimal performance indexes and optimal solutions of linear programs Eq. (29). From Eq. (22),(29) it is also easy to verify that $\sigma(j)_l^i > 0$ indicates that $x(j)_l^i \in S(G, \epsilon(j)_l^i w)$ is outside the ith face of $Q_j(G, \lambda w \epsilon(j)_l^i)$, Eq. (15), which concludes the proof. □

4. CONSTRUCTION OF ROBUSTLY λ-CONTRACTIVE SETS

Consider a closed convex polyhedral set Ω with nonempty intersection with the region of nonlinear behavior of system in Eq. (5). From Definition 1, it is immediate to verify that the family of all robustly λ-contractive polyhedral sets w.r.t. system, Eq. (5), contained in Ω, is closed under the operation of union. Consequently, this family must have a closed supremal element, formed by the union of all its members. However, due to the nonlinear nature of the saturating control law, Eq. (4), convexity of one-step robustly admissible set, Eq. (11), and closure under the convex hull of union cannot be assured. Consequently, convexity of the supremal element cannot also be generally assured. Considering these drawbacks, based on Proposition 1 and Corollary 1, an heuristic non homothetic expansion approach is proposed for constructing a convex robustly λ-contractive polyhedral set with nonempty intersection with the region of nonlinear behavior of system in Eq. (5). In general terms, the following task is executed recursively: given a convex robustly λ-contractive set $S(G_a, w_a)$ and its homothetically expanded $S(G_a, \delta w_a)$, construct a robustly λ-contractive set $S(G, w)$ such that $S(G_a, w_a) \subset S(G, w) \subset S(G_a, \delta w_a)$. For constructing $S(G, w)$, start with $S(G, w) = S(G_a, \delta w_a)$ and recursively intersect $S(G, w)$ with cutting-planes related to faces of one-step robustly admissible set $Q_j(G, \epsilon\lambda w)$, Eq. (15), which do not satisfy Corollary 1. See Procedure 1 in Appendix for a detailed description.

5. NUMERICAL EXAMPLE

Consider the following uncertain system with saturating feedback control law:

$$x(k+1) = Ax(k) + Bu(k)$$

$$\begin{aligned} -\check{u} \leq u \leq \hat{u} \\ u(k) = sat(Fx(k)) \end{aligned} \tag{30}$$

$$A = \begin{bmatrix} 0.8 + q_1 & 0.5 \\ -0.4 & 1.2 \end{bmatrix} \; ; \; B = \begin{bmatrix} 0 \\ 1.0 - q_2 \end{bmatrix} \; ; \; 0 \leq q_1, q_2 \leq .1$$

$$\check{u} = \hat{u} = 7.0 \; ; \; F = \begin{bmatrix} 0.2888 & -1.8350 \end{bmatrix}$$

Assume a contraction rate $\lambda = .998$. $S(G_l, w_l)$, the supremal λ-contractive polyhedral set w.r.t. system, Eq. (30), contained in its region of linear behavior, is given by:

$$G_l = \begin{bmatrix} 0.2888 & -1.8350 \\ 0.4351 & 1.3096 \\ -0.2888 & 1.8350 \\ -0.4351 & -1.3096 \end{bmatrix} \; ; \; w_l = \begin{bmatrix} 7.0 \\ 6.9860 \\ 7.0 \\ 6.9860 \end{bmatrix} \tag{31}$$

Figure 1 presents the following λ-contractive sets w.r.t. system, Eq. (30), with contraction rate $\lambda = .998$:

$S(G_l, w_l)$: supremal robustly λ-contractive polyhedral set w.r.t. system, Eq. (30), contained in its region of linear behavior (Dórea & Hennet, 1996);

$S(G, w)$: robustly λ-contractive polyhedral set w.r.t. system, Eq. (30), constructed by Procedure 1 in Appendix, with parameters $\delta = 1.5$, $\delta_m = 1.1$ and initial set $S(G_l, w_l)$;

$S(G_p, w_p)$: λ-contractive polyhedral set w.r.t. system, Eq. (30), considering system parameters perfectly known, $(q_1 = 0, q_2 = 0)$, constructed by Procedure 1 in Appendix, with parameters $\delta = 1.5$, $\delta_m = 1.1$.

It can be verified, in Fig. 1(A), that polyhedron $S(G, w)$ is much larger than polyhedron of depart $S(G_l, w_l)$ in the region of linear behavior, showing the effectiveness of Procedure 1 in the constrution of a robustly λ-contractive polyhedral set over the region of nonlinear behavior of a closed loop system with saturating controls. As expected, $S(G, w)$ being robust w.r.t model uncertainties, is a subset of $S(G_p, w_p)$ which assumes a perfectly known model. Figure 1(B) shows a trajectory of uncertain closed-loop system, Eq. (30), starting from an extreme point of $S(G, w)$.

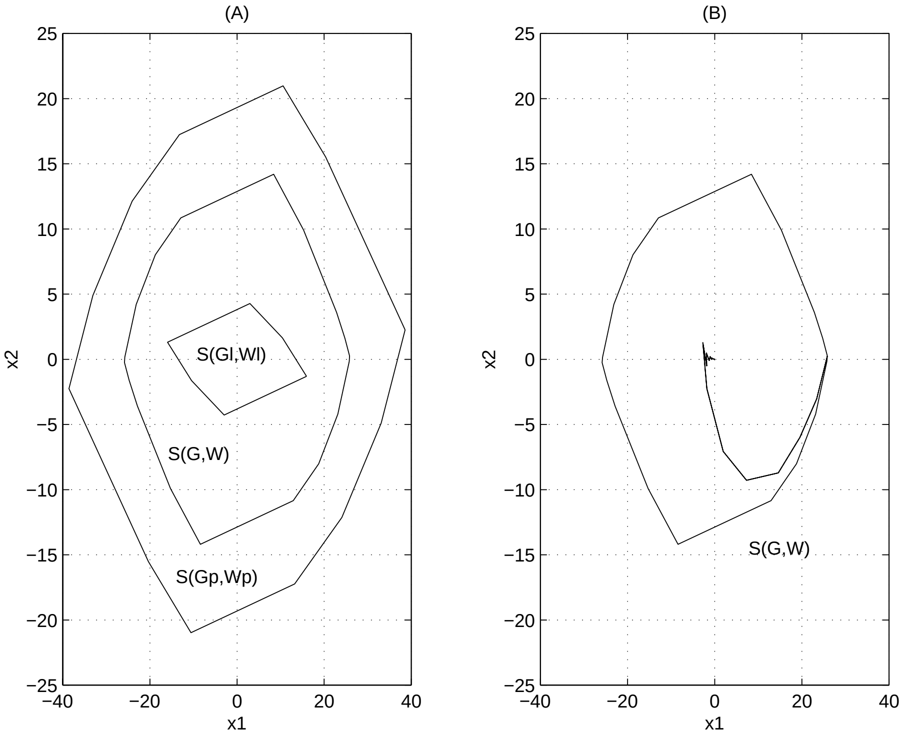

Figure 1: λ-contractive polyhedra

6. CONCLUSION

New necessary and sufficient conditions have been proposed for compact polyhedral sets be robustly λ-contractive w.r.t. discrete-time systems with uncertain domains defined by compact polytopes. The proposed conditions are based on piecewise linear model related to the saturation regions of the feedback law and use facial description of the polyhedral set combined with extreme points of uncertain domain. Using efficient linear programming formulation of these necessary and sufficient conditions, an effective non homothetic expansion procedure has been proposed for construction of robustly λ-contractive sets with nonempty intersection with regions of closed-loop nonlinear behavior of uncertain systems. The procedure starts with the supremal robustly λ-contractive polyhedral set contained in the region of linear behavior of the uncertain system and progressively expands it over the region of nonlinear behavior. The effectiveness of proposed procedure is due to following features: non conservative and computationally efficient λ-contractivity test; judicious avoidance, during the expansion process, of intersections of convex polyhedron in construction with the non convex boundaries of its one-step admissible set; periodical elimination of redundant inequalities in the facial description of polyhedron being constructed.

REFERENCES

Blanchini, F., 1994, Ultimate Boundedness Control for Uncertain Discrete-Time Systems Via Set-Induced Lyapunov Functions, IEEE Transactions on Automatic Control, Vol. 39, No. 2, pp. 428-433.

Dorea, C.E.T. & Hennet, J.C., 1996, Computation of Maximal Admissible Sets of Constrained Linear Systems, Proceedings of 4th IEEE Mediterranean Symposium on New Directions on Control and Automation, Maleme, Greece.

Gilbert, E. G. & Tan, K.T., 1991, Linear Systems with State and Control Constraints: The Theory and Application of Maximal Output Admissible Sets, IEEE Transactions on Automatic Control, Vol. 36, No. 9, pp.1008-1019.

Hennet, J.C., 1989, Une Extension du Lemme de Farkas et son Application au Problème de Regulation Linéaire sous Contraintes, C.R. Acad. Sci. Paris, t.308, Série I, pp. 415-419.

Milani, B.E.A. & Carvalho, A.N., 1995, Robust Linear Regulator Design for Discrete-Time Systems Under Polyhedral Constraints, Automatica, Vol. 31, No. 10, pp. 1489-1493.

Milham, C.B., 1976, Fast Feasibility Methods for Linear Programming, OPSEARCH, Vol. 13, No. 3-4, pp. 198-204.

Silva Jr., J.M.G. & Tarbouriech, S., 1997, Polyhedral Regions of Local Asymptotic Stability for Discrete-Time Linear Systems with Saturating Controls, Proceedings of the 36th Conference on Decision and Control, San Diego CA, USA.

Vassilaki, M., Hennet, J.C. & Bitsoris, G., 1988, Feedback Control of Linear Discrete-Time Systems under State and Control Constraints, International Journal of Control, Vol. 47, No. 6, pp. 1727-1735.

Verriest, E.I. & Pajunen, C.A., 1996, Quadratically Saturated Regulator for Constrained Linear Systems, IEEE Transactions on Automatic Control, Vol. 41, No. 7, pp. 992-995.

APPENDIX

Procedure 1: Construction of a robustly λ-contractive set $S(G, w)$, Eq. (10), w.r.t. system, Eq. (5), involving its regions of nonlinear behavior.

step 1 - Initialization

Choose parameters:

λ - contraction rate

$\delta > 1$ - expansion coefficient

$\delta_m < \delta$ - minimum expansion coefficient

Construct the supremal robustly λ-contractive set $S(G, w)$ contained in region of linear behavior $S_1(R_1, d_1)$ (Blanchini, 1994), (Dórea & Hennet, 1996).

step 2

Eliminate redundant inequalities in $S(G, w)$ (Milham, 1976).

Set: $G_a = G$

step 3 - homothetic expansion

Set: $w_a = w \quad ; \quad w = \delta w_a$

step 4

Check if $S(G, w)$ is robustly λ-contractive **(Routine 1.1)**

If $Answer = Yes$: $S(G, w)$ is λ-contractive, return to step 3.

step 5 - Construction of a robustly λ-contractive set $S(G, w)$ such that

$$S(G_a, \delta_m w_a) \subset S(G, w) \subset S(G_a, \delta w_a)$$

step 5.1 - Define the cutting plane $S(\tilde{g}, \tilde{w})$ to $S(G, w)$ related to j^*, l^*, i^* identified by Routine 1.1

$$\tilde{g} = g^{i^*} A_{j^*}^{l^*} \;\; ; \;\; \tilde{w} = \lambda w_{i^*} - g^{i^*} p_{j^*}^{l^*}$$

step 5.2

Check if $S(G_a, \delta_m w_a) \subset S(\tilde{g}, \tilde{w})$ (Hennet, 1989)

If $Answer = No$: $S(G_a, w_a)$ is the desired robustly λ-contractive set. **Stop**

step 5.3 - Construction of $S(\tilde{g}, \tilde{w}) \cap S(G, w)$

$$G = \begin{bmatrix} G \\ \tilde{g} \end{bmatrix} \;\; ; \;\; w = \begin{bmatrix} w \\ \tilde{w} \end{bmatrix}$$

step 5.4

Check if $S(G, w)$ is robustly λ-contractive **(Routine 1.1)**

If $Answer = Yes$

Then: $S(G, w)$ is robustly λ-contractive. Return to step 2.

Otherwise: return to step 5.1

Routine 1.1: Check if $S(G, w)$ is robustly λ-contractive using Corollary 1.

- For $j = 1 : 3^m$,

  For $l = 1 : s$, $i = 1 : r$, solve the linear programs:

$$\sigma(j)_l^i = \max_{x,\epsilon} g^i A_j^l x - \lambda w_i \epsilon + g^i p_j^l$$

$$\begin{aligned} Gx - w\epsilon &\leq 0 \\ R_j x &\leq d_j \\ 0 \leq \epsilon &\leq 1 \end{aligned}$$

Set

$$\sigma_j = \max_{l,i} \sigma(j)_l^i$$

$$(l_j, i_j) = arg \max_{l,i} \sigma(j)_l^i$$

$$\epsilon_j = \epsilon(j)_{l_j}^{i_j}$$

- If:

$$\max_j \{\sigma_j\} \leq 0$$

Then: set $Answer = Yes$

Otherwise: set $Answer = No$ and identify the inner saturation region not satisfying Corollary 1:

$$j^* = arg \min_j \epsilon_j \quad s.t. \quad \sigma_j > 0$$

$$l^* = l_{j^*} \ ; \ i^* = i_{j^*}$$

Some remarks about Procedure 1 are opportune:

Selection of the inner saturation region by Routine 1.1 in steps 4, 5.4 and the inclusion check in step 5.2, are an attempt to avoid cutting-planes intersecting $S(G_a, \delta_m w_a)$ due to the non convexity of $Q(G, \epsilon w)$. See Fig. 2;

Elimination of redundant inequalities in step 2 is strongly recommended, not only to obtain a concise representation of $S(G, w)$, but also for the overall computational effectiveness of the procedure;

If convenient, any robustly λ-contractive convex polyhedral set can be used as initial set in step 1.

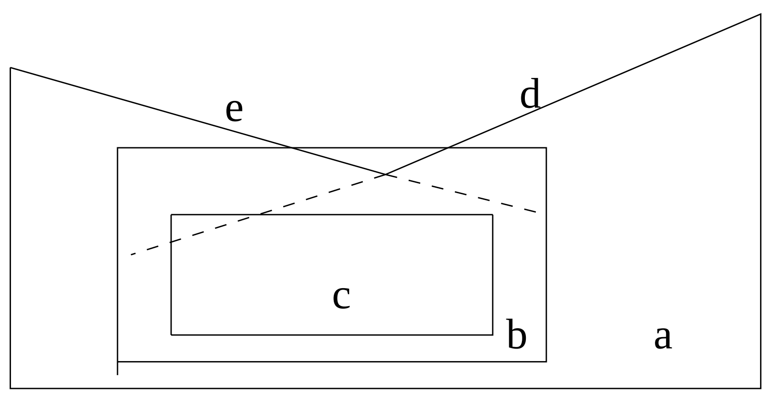

Figure 2: a - $Q(G, \epsilon w)$ b - $S(G, \epsilon w)$ c - $S(G_a, \epsilon \delta_m w)$
d - not acceptable cutting-plane e - $S(\tilde{g}, \epsilon \tilde{w})$